Time of the first scintillation photon

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December 18, 2019

Scintillation time distribution

- ***Scintillation photons have an exponential probability density distribution in time.***
- ***A collection of n scintillation photons, each independent, will form a time series. Such a series will form a signal from a photo-multiplier tube.***
- ***For our purposes here, we will assume that the detector has very fast time response, and no noise. When we measure the time of the pulse, it is essentially the time of the first photon in the time series.***
- ***The question: given that n scintillation photons have an exponential time series, what is the probability density function in time of the first photon ?***
- ***The first scintillation photon is obviously one with the minimum time. Therefore the question can be reframed on the next slide.***

X_i are n random variables, each drawn from a PDF given by $P_X(x)$

$Z = Max(X_1, X_2, ..., X_n)$, and $T = Min(X_1, X_2, ..., X_n)$

It is sometimes easier to calculate cumulative proability and then differentiate to get the density.

Let $\mathrm{F}_X(x) =$ the cumulative probability that $(X \le x) = \int_0^x P_X(x)\,dx$

$1 - F_X(x) =$ the cumulative probability that $(X \ge x)$

It is obvious that

$F_Z(z) =$ Probability($X_1 \le z$ and $X_2 \le z$ and ... $X_n \le z$)

$$= \prod_{i=1}^{n} F_{X_i}(z)$$

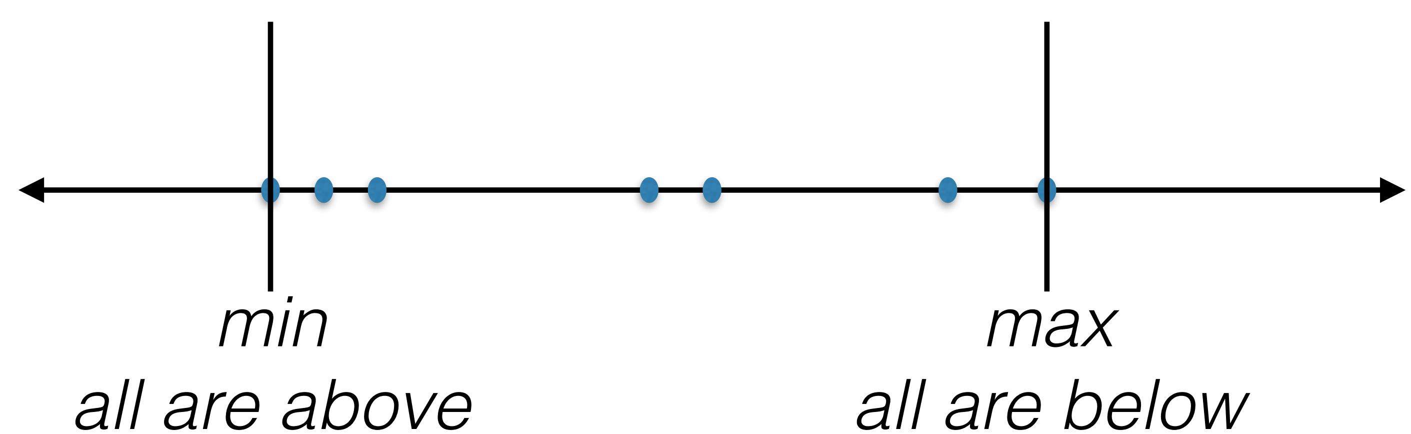
In words: the probability that the maximum of the series of numbers is less than a value is the probability that all numbers are less than that value. Since all the numbers are independent this is just

$F_Z(z) = (F_X(z))^n$

We now differentiate this to get

$P_Z(z) = nP_X(z)\left[F_X(z)\right]^{n-1}$

similarly $P_T(t) = nP_X(t)[1 - F_X(t)]^{n-1}$



Now let's imagine scintillation photons with time parameter of τ

Keeping the same nomenclature: $\mathrm{P}_X(x) = \frac{1}{\tau} e^{-x/\tau}$

$F_X(x) = [1 - e^{-x/\tau}]$

The PDF for the minimum of n photoelectrons is

$\mathrm{P}_T(t) = nP_X(t)\left[1 - (1 - e^{-t/\tau})\right]^{n-1}$

$P_T(t) = n\frac{1}{\tau} e^{-t/\tau}[e^{-t/\tau}]^{n-1}$

$P_T(t) = \frac{n}{\tau} e^{-nt/\tau}$ .... for $n \geq 0,\ t \geq 0$

The minimum also has an exponential PDF with much smaller time contant.

Time resolution is given by the square root of variance of the exponential

$\sigma = \frac{\tau}{n}$

Counter to the typical instinct of physicists: the time resolution improves linearly with the number of photo-electrons. This is very important !

For liquid argon time constant of 6 ns, pulses of >6 photoelectrons are needed to get 1 ns intrinsic time resolution.

Let's now imagine that the light is coming from a long track and therefore the pulse width is mostly due to the extent of the event. We start with a uniform flat distribution.

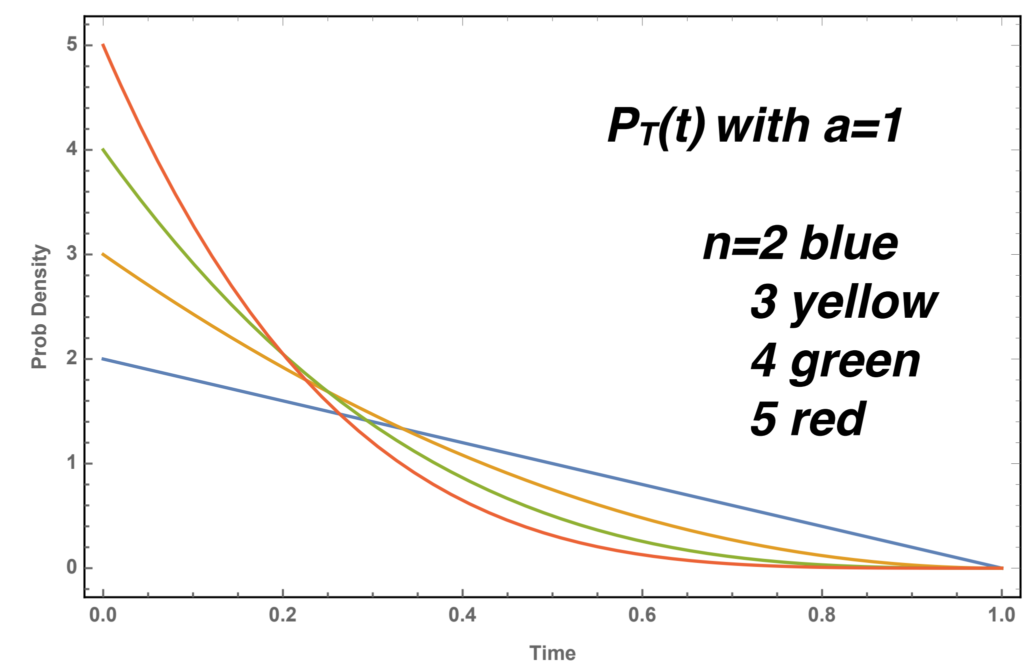
$P_X(x) = 1/a \qquad$ for $0 < x < a$, and 0 otherwise

$F_X(x) = x/a \qquad$ for $0 < x < a$, and 0 otherwise

For the first photon coming from a uniform PDF.

$P_T(t) = nP_X(t)[1 - t/a]^{n-1}$

$\mathrm{P}_T(t) = \dfrac{n}{a}[1 - t/a]^{n-1} \qquad$ for $0 < t < a$, and 0 otherwise.



The mean of this is $\langle T \rangle = \dfrac{a}{n+1}$

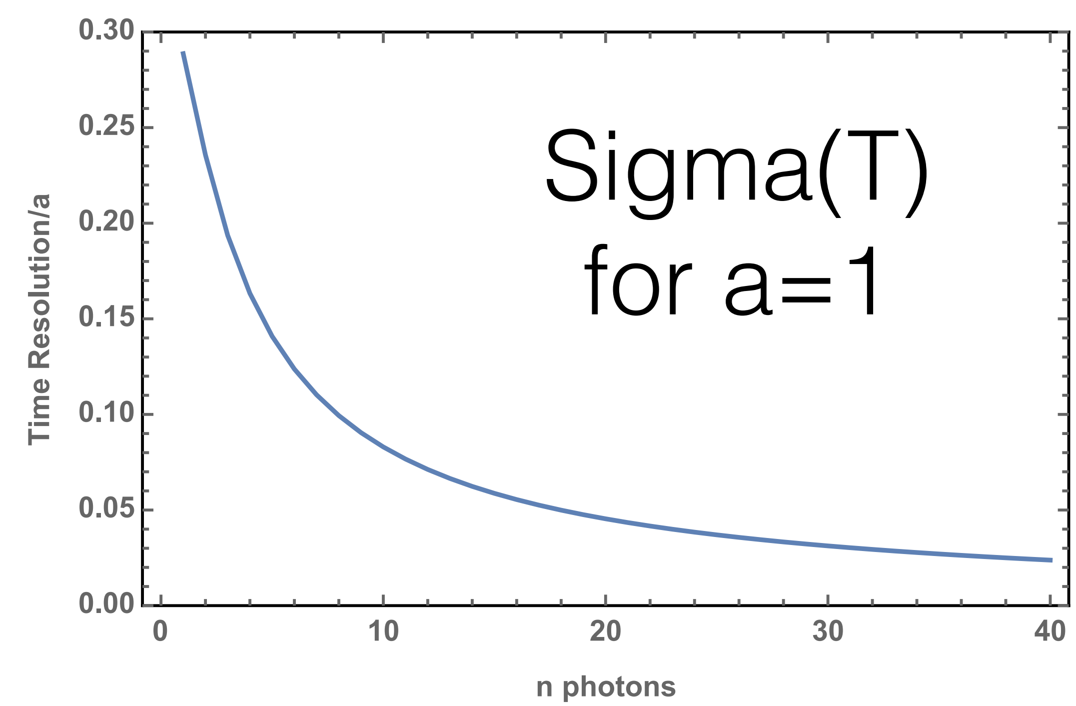
and the variance is $Var[T] = a^2\left[\dfrac{2n!}{(n+2)!} - \dfrac{1}{(n+1)^2}\right]$

We will plot the Square root[Var] for $a = 1$ versus n

Notice that for $n = 1$ we recover the usual $\text{var} = a^2/12$

As n increases the resolution does not improve linearly.

Suppose the event size is 10 ns in time extent, then to get 1 ns timing will require a lot more than 20 photo-electrons.



Let's now extend the analysis to calculate the expected resolution including the fluctuation in the number of photons n.

For the probability density $P_T(t)$, we have assumed a constant number of photons $(n \geq 1)$. In particular, for $n = 0$, the density is undefined. And so we have to be careful, when working with a fluctuating n. Let's first denote $P_T(t) \rightarrow P_{T,n}(t)$ to indicate its dependence on n

We now assume that n has Poisson density function $P_\lambda(n) = e^{-\lambda} \dfrac{\lambda^n}{n!}$

The sample space (list of all outcomes) now has two components: there is a chance that we obtain no time for events with no photons, and of obtaining a definite $t \geq 0$.

If there are non-zero photon then we must weigh each $P_{T,n}(t)$ by the probability of obtaining n and sum up the result.

$$\text{Probability(t or no-t)} = \left[\begin{array}{ll} e^{-\lambda} & \text{for no definite event time.} \\ & \displaystyle\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times P_{T,n}(t) \end{array} \right.$$

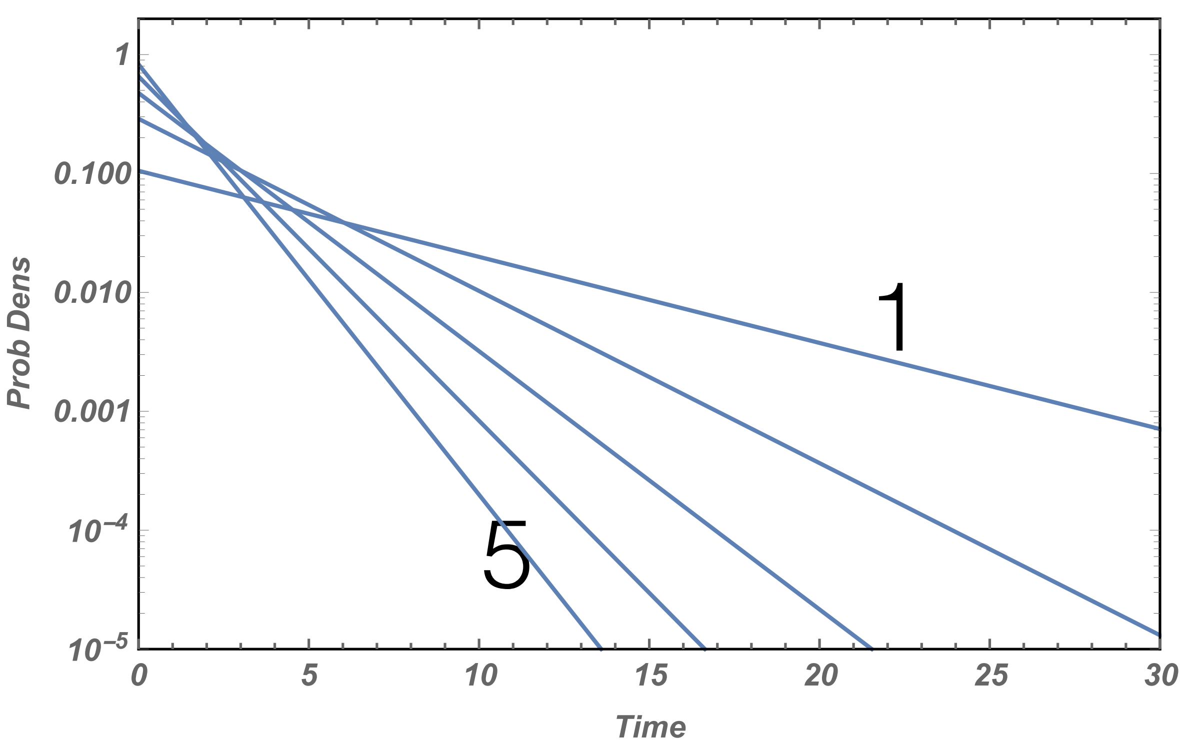
Notice that the integral of the second term is the probability: $(1 - e^{-\lambda})$ of obtaining a definite time or having at least one photon.

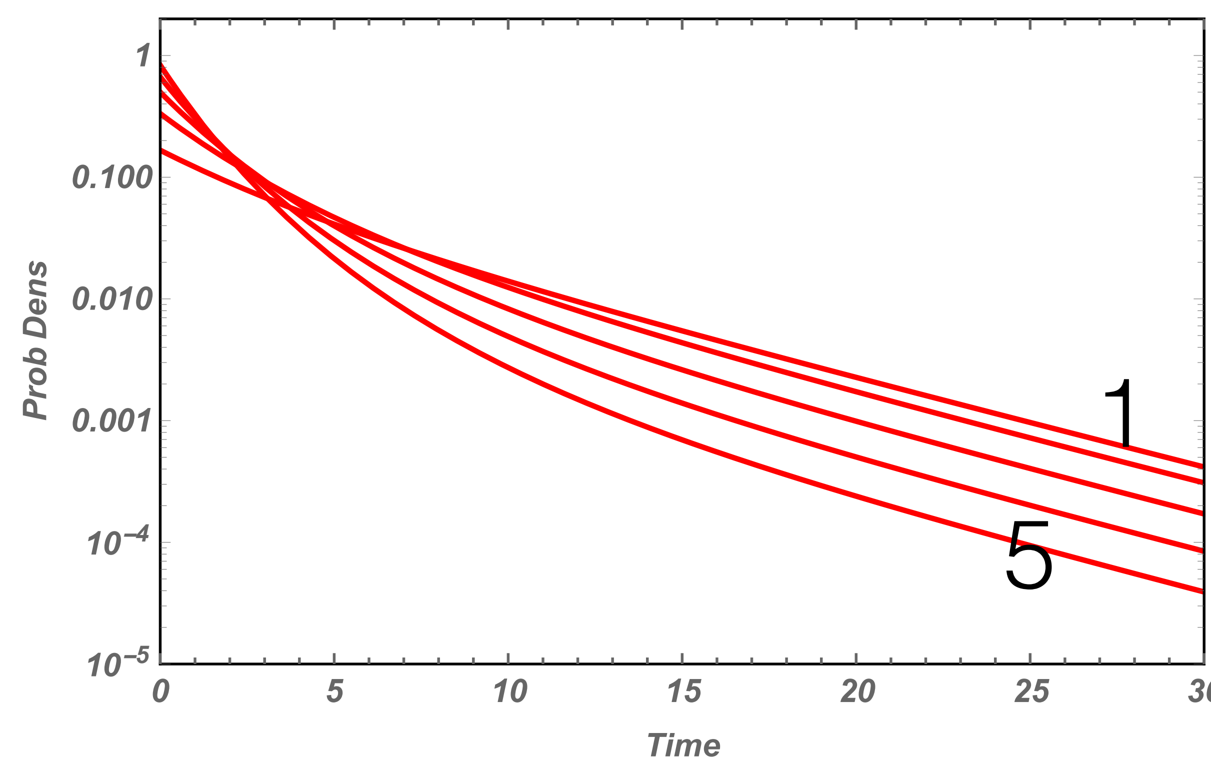
We will now compare the time distribution with and without fluctuation of the number of detected photons. But only for the case where at least 1 photon is detected. In case of no fluctuation, we plot $P_{T,n}(t)$ where n is replaced by the mean number of expected photons. We need to renormalize it for the probability of nonzero photons. We limit ourselves to exponential PDFs for single photons.

$$P_{nofluct}(t) = (\text{P\_of\_n} \neq 0) \times P_{T,n}(t) = \frac{\lambda}{\tau} e^{-\lambda t/\tau} \times (1 - e^{-\lambda})$$

$$P_R(t) = \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times P_{T,n}(t)$$

We set $\tau = 6$ and λ=1, 2, 3, 5, and sum upto $n = 100$. The curves with λ=1, and 5 are marked





When fluctuations are present a tail develops in time

The mean and variance for $P_{T,n}(t) = \frac{n}{\tau} e^{-nt/\tau}$ are given by

$M[t] = \tau / n$ and $\mathrm{Var[t]} = \tau^2 / n^2$

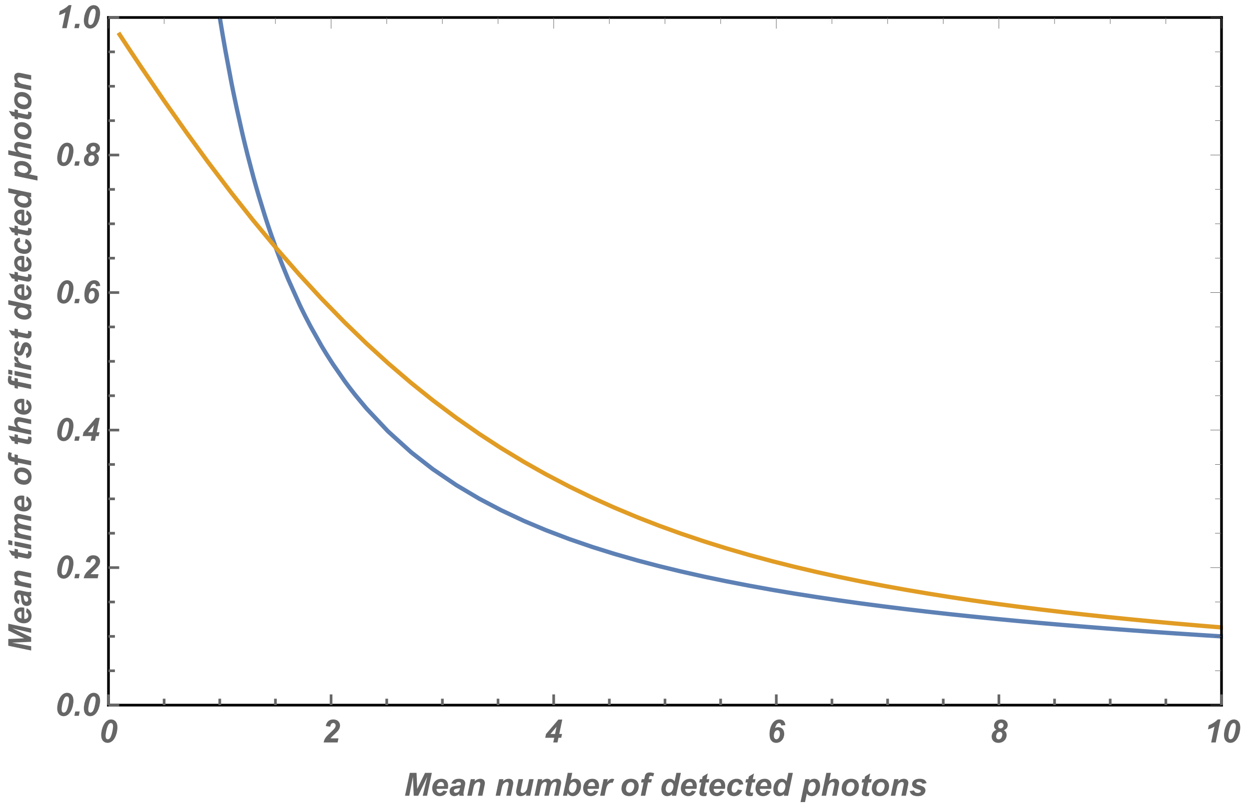
For $\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times P_{T,n}(t)$, we need to be a bit careful; recall that we have to normalize the PDF by its integral first. And use the definition of of mean as $E[\![t]\!]$ and variance as $E[\![t^2]\!] - (E[\![t]\!])^2$

$$\mathrm{M[t]} = \tau \times \frac{\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times \frac{1}{n}}{1 - e^{-\lambda}}$$

$$\mathrm{Var[t]} = 2\tau^2 \times \frac{\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times \frac{1}{n^2}}{1 - e^{-\lambda}} - \tau^2 \times \left[\frac{\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times \frac{1}{n}}{1 - e^{-\lambda}} \right]^2$$

$$\mathrm{Var[t]} = \frac{\tau^2}{(1 - e^{-\lambda})} \times \left[2\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times \frac{1}{n^2} - \frac{1}{1 - e^{-\lambda}} \left(\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times \frac{1}{n} \right)^2 \right]$$

We will now plot this as a function of λ for τ=1.

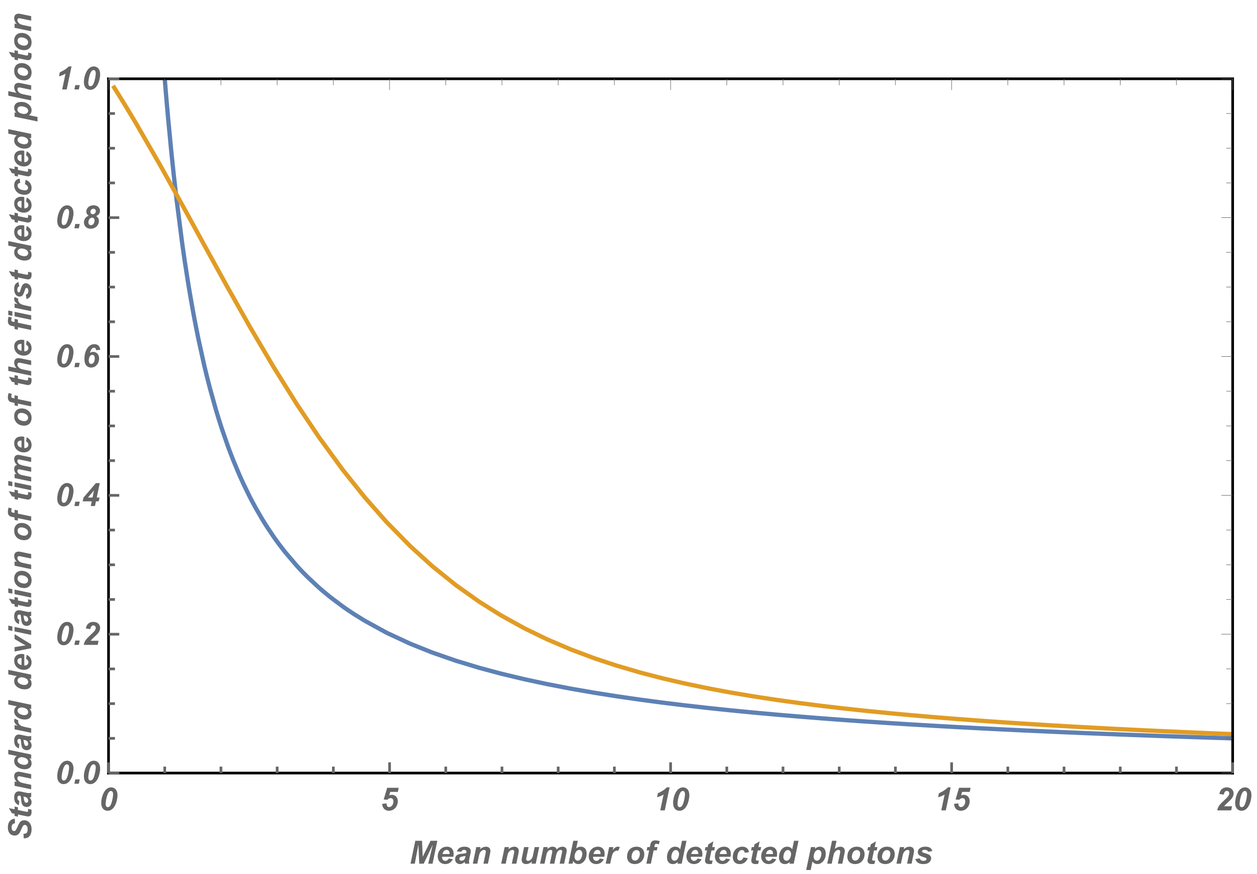


The exponential parameter is τ=1

The blue curve is for no fluctuations, and so $mean = \tau / n$.

(For the blue curve, $n > 0$ has to be integer)

The orange curve is for Poisson fluctuation.



This is the square-root of the Variance on the first photon time.

The exponential parameter is τ=1

The blue curve is for no fluctuations, and so std dev. $\sigma = \tau / n$.

(For the blue curve, $n > 0$ has to be integer)

The orange curve is for Poisson fluctuation.

The effect of the long tail is clear on the standard deviation when mean number of photons is >1.

Incidently, this is only for detected photons, and therefore the standard deviation has to remain $<\tau$ even when the mean photons is very small.

As a final comment we note that there are two additional processes that could cause variation in the first photon time. These are due to dark current from the photo-sensor or electronics noise in the front end. Either of these could cause fluctuations (for example voltage fluctuations) that alter the time at which the signal goes above threshold. We can simply account for both of these effects by introducing an additional Gaussian smearing with a parameter, σ.

From our previous discussion, a convolution of Gaussian with an exponential yields a "Exponentially modified Gaussian" which we define here.

The Normal PDF

$$N(x:\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\ \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Exponential modified Normal PDF

$$E_N(x:\mu,\sigma^2,\lambda)=\frac{\lambda}{2}e^{\frac{\lambda^2\sigma^2}{2}}e^{-\lambda(x-\mu)}Erfc\left[\frac{1}{\sqrt{2}}\left(\lambda\sigma-\frac{x-\mu}{\sigma}\right)\right]$$

μ: mean of the Gaussian

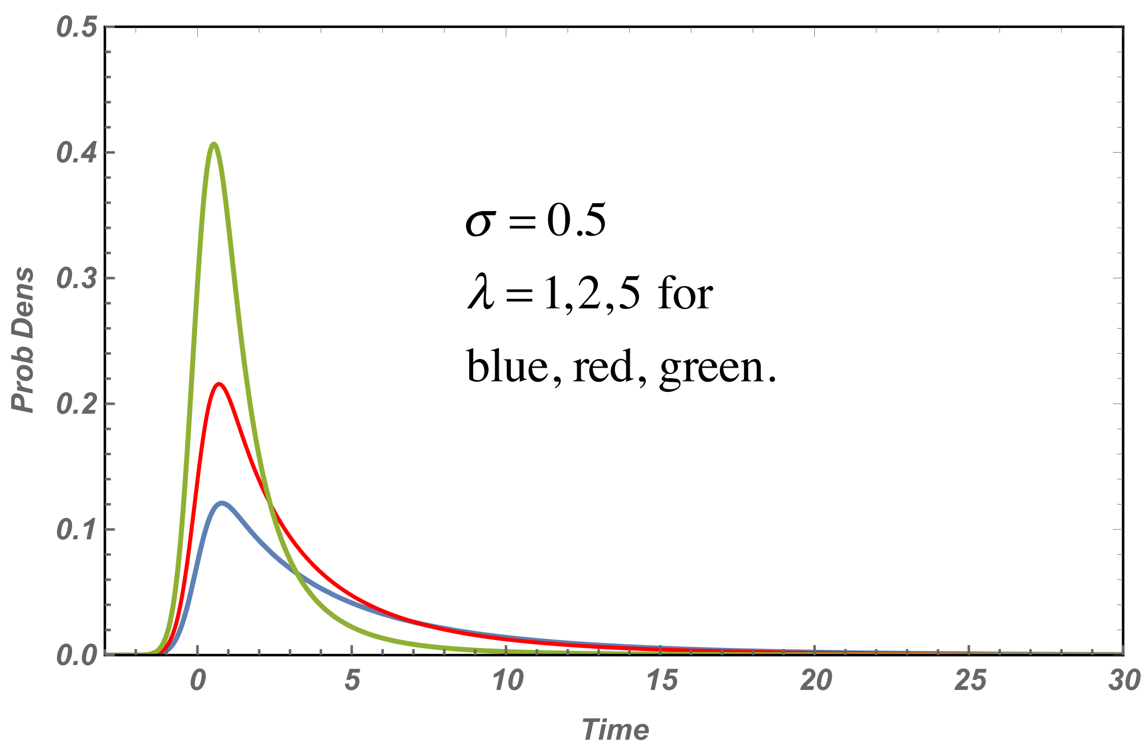
σ^2: variance of the Gaussian

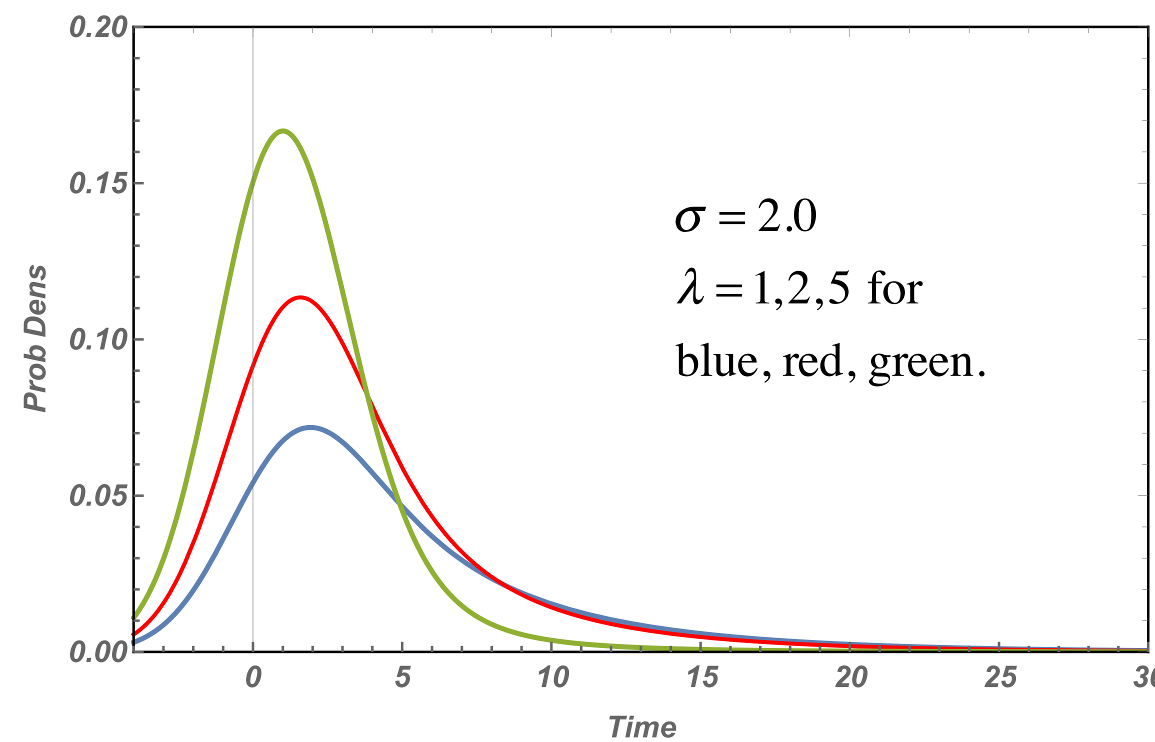
λ: exponential decay parameter

The answer for the distribution of first photon time including the effects of noise is now given by

$$\text{Probability(t or no-t)} = \left[\begin{array}{ll} e^{-\lambda} & \text{for no definite event time.} \\ \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times E_N(t:0,\sigma^2,n/\tau) & \text{when there is a time} \end{array} \right.$$

We plot this for some examples. We set τ=6; σ=0.5, 2 and λ=1, 2, 5





Notice how there is a tail for low number of photons. We leave it to the reader to calculate the mean and variance of these.

conclusion

- ***We have calculated the probability density function for measurement of time from a finite number of detected photons.***

- ***Each photon is assumed to come independently from scintillation process. We focused on an exponential PDF for the photon process.***

- ***The first photon was assumed to provide the time.***

- ***We show that the PDF for the time measurement has a significant positive tail. This tail develops because of fluctuations in the detected number of photons some of which could come late.***

- ***A detailed calculation can be performed to provide an expression for the time PDF, its mean, and variance.***